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On the composition of gauge structures

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Abstract. A formulation for a non-trivial composition of two classical gauge structures is given: two parent gauge structures of a common base space are synthesized so as to obtain a daughter structure which is fundamental by itself. The model is based on a pair of related connections that take their values in the product space of the corresponding Lie algebras. The curvature, covariant exterior derivatives and associated structural identities all get contributions from both gauge groups. The various induced structures are classified into those whose composition is given just by trivial means, and those which possess an irreducible nature. The pure irreducible piece, in particular, generates a complete super-space of ghosts with an attendant set of super-BRST variation laws, both of which are purely of a geometrical origin.

1. Introduction

It has generally been accepted that composite classical gauge structures of a non-supersymmetric nature are constructed via trivial splicing. Indeed, the formal description of principal bundles could thus easily be extended to include bundles whose fibres are product spaces, with a different gauge-group acting independently on each factor fibre space in the product. Any other composition process breaks this formal scheme within which different gauge structures, introduced on the same underlying manifold, are mutually transparent to one another and might somehow be correlated only by artificial means.

This paper suggests a new type of gauge theory which results from a non-trivial composition of two genuine gauge structures. In this theory the geometry is not split, even though the bundle is. It is first developed for splices of fibre bundles which comply with some severe algebraic requirements realizing very clear-cut principles. Extending the theory to incorporate a wider class of gauge structures is shown to be straightforward, provided some soft structural requirements at the level of the algebra are fulfilled. Within the extended framework, the single-fibre structures and the process of trivial splicing appear only as sub-sectors in a much larger and comprehensive construction.

In what follows, things are formulated in a geometrical setting by using a coordinate-free language. After a short prelude which serves mainly to fix the notation, we define the notion of a foliar complex, exhibiting its rich geometrical content and 'peculiar' algebraic structure. In particular, the covariant bundle operators and the associated structural identities are strictly recovered. We then present an algebraic interpretation, in which more insight is gained and from which geometric links are drawn. The invariance of the curvature with respect to local translations in the spaces of connections is later examined and it is found that it directly implies a whole super-BRST sector. Finally, upon removing some of the initial constraining requirements, the theory is shown to accommodate many types of gauge structures.

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2. Foliar complexes on spliced fibre bundles

Consider a smooth manifold M of n dimensions, defined over a field K. Let T_xM be the space tangent to M at a point $x \in M$, and let $T_x^\star M$ denote the space of all linear functionals on T_xM. One usually constructs the *classical Grassmann bundle* $\wedge TM$, the bundle whose sections are antisymmetric tensor fields on M, and the counterpart co-bundle $\wedge T^\star M$, the one whose sections are K-valued differential forms, by gluing together the Grassmann algebras associated with each and every (co-) tangent space at each and every point of M,

$$\wedge\, TM := \bigcup_{x\in M}\left(\bigoplus_{p=1}^{n}\bigwedge\nolimits^{p} T_xM\right) \qquad \wedge\, T^\star M := \bigcup_{x\in M}\left(\bigoplus_{p=0}^{n}\bigwedge\nolimits^{p} T_x^\star M\right) \tag{1}$$

where the union and the direct sum are commutable. In particular, the first summand of $\wedge TM$ above is the tangent bundle, the zeroth summand of $\wedge T^\star M$ is the space of all functions on M, and $T^\star M$ is the co-tangent bundle of one-forms. For any α, β differential forms $\in \wedge T^\star M$, one defines their graded brackets ('commutator') by $[\alpha, \beta] := \alpha \wedge \beta - \left(-1\right)^{pq} \beta \wedge \alpha$, where p is the form-degree of α and q is that of β. In addition, the underlying manifold is naturally equipped with an exterior derivative d which maps p-graded objects into $\left(p+1\right)$-graded ones. Being a co-boundary operator over $\wedge T^\star M$, it satisfies $d \circ d = 0$ on forms, and obeys the graded Leibnitz rule with respect to wedge multiplication: $d\left(\alpha \wedge \beta\right) = d\alpha \wedge \beta + \left(-1\right)^{p} \alpha \wedge d\beta$.

Once a structure group and a representation space are specified, a gauge structure is established. The *classical gauge* is by definition a $\mathrm{GL}\left(n, K\right)$ structure in its fundamental representation, that is, the gauge group is taken to be the group of general linear transformations acting on the fibres of the classical Grassmann bundle. The classical gauge directly decodes many of the geometric properties of M and, with the additional requirement of being invariant under base space diffeomorphisms, provides a framework for n-dimensional theory of general relativity (without spinors). In its very nature the construction is self-symmetric: base-space components and fibre components are completely interchangeable, and quite often mutually contracted.

Consider next two finite-dimensional representation spaces V and U, of two respective distinct Lie groups G_V and G_U, with respective dimensions n_V and n_U, each of which meet the following stringent requirement: each set of generators of the two forming groups, when carrying an appropriate representation in the corresponding representation space, must *close* under anticommutation relations as well [1]. The existence of these closure requirements, as well as their formal form, crucially depends on the particular representation that is chosen. For example, it may be satisfied in a fundamental representation, but may not exist, or exist in a different form, for higher representations. In particular V and U are taken to be the fibres of two associated bundles having a common base space, a smooth manifold M of n dimensions. Because the two associated bundles VM and UM are smooth by themselves, the so-called *local leaves* $S_x := V_x \times U_x$ can be smoothly glued to form a spliced fibre bundle $SM := \bigcup_{x\in M} S_x$ composed of two structures. In a similar manner, one may take copies of the product space $V_x \times U_x$, and form two-structure bundles of higher dimensional leaves.

Once the splice is globally formed, x-dependent frames are assigned to each and every local leaf. The smoothly-glued local frames draw sections (frame fields) with which every geometric object $\in SM$ can be described. In what follows we shall draw the attention only to those geometric differential forms on which G_V and G_U act linearly, and in the same manner. Termed *leaf-valued forms*, they transform like leaf vectors or leaf rank-r tensors with respect to the combined action of $G_V \times G_U$, but we shall not restrict ourselves only to combined actions. Other types of differential forms with which we are much interested are forms of the structural kind, taking their values in the *product-space* of the corresponding two Lie algebras, $\mathrm{Lie}\, G_V \otimes \mathrm{Lie}\, G_U$. The whole of the leaf-valued differential forms and the whole of the $\left(\mathrm{Lie} \otimes \mathrm{Lie}\right)$-valued ones constitute a totality of forms Γ_{FC}, each of its members represented irreducibly and non-trivially on both fibres. The corresponding *foliar complex* (FC) of irreducible bundle objects will just be associated with a particular geometric infrastructure of gauge in which projections on fibres, and the global process of identity reduction, are no longer natural attributes of the splice. In fact, the foliar formation accommodates two group structures in a non-contractable fashion.

Comment. The obligatory demand for closure of the algebras under anticommutation can be much softened, for example, by considering cases where anticommutators of group generators in a representation *also* include terms proportional to the identity. This type of extension necessarily generalizes the concept of a foliar complex. For pedagogical reasons, however, we postpone any of this to section 6.

3. The geometry associated with a foliar complex

Let us begin by proposing the following.

Proposition. The two-form structure

$$\begin{aligned} R_{FC} &:= d\left(\varphi + \omega\right) + \left(\varphi + \omega\right) \wedge \left(\varphi + \omega\right) \\ &= d\varphi + \varphi \wedge \varphi + \varphi \wedge \omega + \omega \wedge \varphi + \omega \wedge \omega + d\omega \end{aligned} \tag{2}$$

is a linear curvature $\in$ FC provided that the pair of connection one-forms $\left(\omega, \varphi\right)$ take their values in $\mathrm{Lie}\, G_V \otimes \mathrm{Lie}\, G_U$ and obey the set of transformation laws

$$\begin{aligned} &\forall\, v \in G_V: \quad \omega \mapsto v\omega v^{-1} \quad \varphi \mapsto v\left(\varphi + d\right)v^{-1} \\ &\forall\, u \in G_U: \quad \varphi \mapsto u\varphi u^{-1} \quad \omega \mapsto u\left(\omega + d\right)u^{-1} \end{aligned} \tag{3}$$

that is, the interleaved fibres interchange the geometric roles played by ω and φ: from the point of view of G_V, φ plays the role of a connection and ω transforms as a tensor. However, from the point of view of G_U, ω plays the role of a connection and φ transforms as a tensor.

Comment. True, $\mathrm{Lie}\, G_V \otimes \mathrm{Lie}\, G_U \neq \mathrm{Lie}\left(G_V \times G_U\right) = \mathrm{Lie}\, G_V \otimes I_U + I_V \otimes \mathrm{Lie}\, G_U$, where I stands for an identity. Please note, however, that $\mathrm{Lie}\, G_V \otimes \mathrm{Lie}\, G_U$ and $\mathrm{Lie}\left(G_V \times G_U\right)$ are both embedded in $\left(\mathrm{Lie}\, G_V + I_V\right) \otimes \left(\mathrm{Lie}\, G_U + I_U\right)$.

Proof. The proof follows two steps: one first shows that the coefficients of the wedge-products in R_{FC} lie in Lie $G_V \otimes$ Lie G_U, and second, that R_{FC} transforms linearly, and in an independent manner, with respect to both gauge groups. Throughout this work we use bare letters to denote anything which is attached to the V-fibre and barred letters to denote anything which is attached to the U-one. Let then the $n_V \times n_U \left(= \dim G_V \times \dim G_U\right)$ tensor products $L_a \otimes \bar{L}_{\bar{a}}$ span a basis for Lie $G_V \otimes$Lie G_U in the leaf representation space $S = V \times U$. In this basis the two foliar connections read $\omega = \omega_{a\bar{a}}\, L^a \otimes \bar{L}^{\bar{a}}$ and $\varphi = \varphi_{a\bar{a}}\, L^a \otimes \bar{L}^{\bar{a}}$. The realization of the commutation relations among the generators must be followed (formally at least) by the indispensable requirement for closure anticommutability:

$$\begin{aligned} &\left[L^a, L^b\right] = f_V^{abc} L_c \qquad &&\left\{L^a, L^b\right\} = g_V^{abc} L_c \\ &\left[\bar{L}^{\bar{a}}, \bar{L}^{\bar{b}}\right] = f_U^{\bar{a}\bar{b}\bar{c}} \bar{L}_{\bar{c}} \qquad &&\left\{\bar{L}^{\bar{a}}, \bar{L}^{\bar{b}}\right\} = g_U^{\bar{a}\bar{b}\bar{c}} \bar{L}_{\bar{c}}\,. \end{aligned} \tag{4}$$

Note: the algebra's *structure constants* f, and the *Clebsch–Gordan coefficients* g of the $V \times V \mapsto V$ and $U \times U \mapsto U$ homomorphisms, are not automatically derived form trace formulae since the associated Cartan metric tensors are usually no longer invertible. Now, the term $\omega \wedge \omega$, say, reads (square brackets over indices stand for antisymmetrization; representation space indices are suppressed)

$$\begin{aligned} \omega \wedge \omega &= \omega_{[\mu|a\bar{a}} L^a \otimes \bar{L}^{\bar{a}} \omega_{\nu]\,b\bar{b}} L^b \otimes \bar{L}^{\bar{b}} \otimes \boldsymbol{e}^\mu \wedge \boldsymbol{e}^\nu \\ &= \omega_{\mu\, a\bar{a}}\, \omega_{\nu\, b\bar{b}} \left[L^a \otimes \bar{L}^{\bar{a}}, L^b \otimes \bar{L}^{\bar{b}}\right] \otimes \boldsymbol{e}^\mu \wedge \boldsymbol{e}^\nu \end{aligned} \tag{5}$$

where the set of $n \left(= \dim M\right)$ co-frame fields $\{\boldsymbol{e}^\mu\}$ span the basis for $T^\star M$. However

$$\left[L^a \otimes \bar{L}^{\bar{a}}, L^b \otimes \bar{L}^{\bar{b}}\right] = L^a L^b \otimes \left[\bar{L}^{\bar{a}}, \bar{L}^{\bar{b}}\right] + \bar{L}^{\bar{a}} \bar{L}^{\bar{b}} \otimes \left[L^a, L^b\right] - \left[L^a, L^b\right] \otimes \left[\bar{L}^{\bar{a}}, \bar{L}^{\bar{b}}\right].$$

The third term of the resulting expansion clearly lies in Lie $G_V \otimes$ Lie G_U. As for the other two terms in the expansion, one should first reconstruct the wedge multiplication and then, only with respect to one of the antisymmetric counterparts, simultaneously re-shuffle summed bare and barred algebra-space indices. As a result, and due to the antisymmetry of the algebra structure constants in their first two indices, the two free pairs of generators respectively convert to anticommutators

$$L^a L^b \to \left\{L^a, L^b\right\} \qquad \bar{L}^{\bar{a}} \bar{L}^{\bar{b}} \to \left\{\bar{L}^{\bar{a}}, \bar{L}^{\bar{b}}\right\}$$

and the closure conditions (4) can then be fully implemented. The product of $\omega \wedge \omega$ finally reads $\frac{1}{2} C_{VU\ c\bar{c}}^{a\bar{a}b\bar{b}}\, \omega_{a\bar{a}} \wedge \omega_{b\bar{b}} L^c \otimes \bar{L}^{\bar{c}}$, where we introduce the *foliar constants*

$$C_{VU}^{a\bar{a}b\bar{b}c\bar{c}} \equiv \tfrac{1}{2}\left(f_V^{abc} g_U^{\bar{a}\bar{b}\bar{c}} + f_U^{\bar{a}\bar{b}\bar{c}} g_V^{abc}\right) - f_V^{abc} f_U^{\bar{a}\bar{b}\bar{c}}$$

characteristic of foliar complexes. The considerations above certainly apply to $\varphi \wedge \varphi$, and moreover, to $[\varphi, \omega] = \varphi \wedge \omega + \omega \wedge \varphi$, always along with the same C_{VU}'s. Therefore

$$R_{FC} = \sum_{\varpi, \varpi' = \varphi, \omega} \left(d\varpi_{c\bar{c}} + \tfrac{1}{2} C_{VU\ c\bar{c}}^{a\bar{a}b\bar{b}} \varpi_{a\bar{a}} \wedge \varpi'_{b\bar{b}}\right) L^c \otimes \bar{L}^{\bar{c}}. \tag{6}$$

This concludes our analysis of the structure of the curvature, which is thus seen to lie in Lie $G_V \otimes$ Lie G_U.

One's second concern corresponds to the transformation properties of R_{FC} $\left(\equiv R\right)$. Consider first the action applied to the curvature by an element v of G_V. Making use of $v dv^{-1} = -\left(dv\right)v^{-1}$ and taking care of the correct signs in successive operations (for example, $d \circ \varphi \wedge \left(\cdot\right) = d\varphi \wedge \left(\cdot\right) - \varphi \wedge d\left(\cdot\right)$) one immediately infers that R can be decomposed into three terms, $d\omega + \varphi \wedge \omega + \omega \wedge \varphi$ and $\omega \wedge \omega$ and $d\varphi + \varphi \wedge \varphi$, each of which transforms linearly in an independent manner. Acting, however, with an element u of G_U, one finds a different decomposition of linearly transformed terms, obtained from the previous one by the interchange $\omega \leftrightarrow \varphi$. Therefore, $\forall\, v \in G_V$, $R \mapsto vRv^{-1}$, and $\forall\, u \in G_U$, $R \mapsto uRu^{-1}$, and because G_V and G_U act in different spaces, we also have $\forall \left(v \times u\right) \in G_V \times G_U$, $R \mapsto \left(v \times u\right) R \left(v \times u\right)^{-1}$. □

The pair of connection one-forms and the associated curvature two-form are structure-type forms $\in \Gamma_{FC}$. They take their values in a space spanned by constant matrices. In what follows we would like to develop the concept of covariant differentiation of leaf-valued forms, sections of the spliced fibre bundle: Let ψ denote a generic leaf p-form, $G_V \times G_U$ vector-valued, or $G_V \times G_U$ tensor-valued of an arbitrary rank. Then, if ψ is vector-valued, so is the quantity $D\psi := d\psi + \left(\omega + \varphi\right) \wedge \psi$ and if ψ is tensor-valued, $D^\star\psi := d\psi + \left(\omega + \varphi\right) \wedge \psi + \left(-1\right)^{\left(p+1\right)} \psi \wedge \left(\omega + \varphi\right)$ is tensor-valued as well. The two operators D and $D^\star$ (whose powers are defined through composition) are structure-preserving differentiations of the foliar complex. They are respectively called the *covariant exterior derivatives* of vector-valued and tensor-valued foliar forms. Note, however, that the application of the graded Leibnitz rule in the context of these exterior derivatives only makes sense for homogeneous multiples of leaf-valued forms. Namely, all forms in the product are one by one leaf-vector-valued or else, one by one leaf-tensor-valued. Otherwise, the resulting derivation will transform non-linearly. It is for exactly this reason that homogeneous compositions are so fundamental to foliar complexes; the non-homogeneous ones do not support covariance.

Having the covariant exterior differentiations in hand, one may redefine the two-form curvature via $[D, D]\,\psi = 2\left(d + \left(\omega + \varphi\right)\wedge\right) \circ \left(d + \left(\omega + \varphi\right)\wedge\right)\psi := 2R \wedge \psi$, or alternatively via $[D^\star, D^\star]\,\psi = 2D^\star \circ D^\star\psi = 2\left(R \wedge \psi - \psi \wedge R\right) := 2\,[R, \psi]$. Now, if one is able to find such a special form $\psi = \Psi$, valued in $\mathrm{Lie}\, G_V \otimes \mathrm{Lie}\, G_U$, then the second definition above is just the Bianchi-like identity for the foliar's *torsion* $T := D^\star\Psi$. Furthermore, by a direct calculation, or by applying the Jacobi identity, $0 = [D^\star, [D^\star, D^\star]]\,\psi = 2D^\star \circ [R, \psi] - 2\,[R, D^\star\psi]$, one arrives at $D^\star R = 0$ which is the Bianchi-like identity for the curvature. These two structural identities in turn imply the Ricci identity $D^\star \circ D^\star T = [R, T]$. Moreover, let us put $\phi := D^\star\psi$. Then, for complexes over infinite-dimensional manifolds (with finite-dimensional leaves), the asymptotic formula $\left(\exp D^\star \circ D^\star\right) \circ \phi = \left(\exp R\right)\phi\left(\exp -R\right)$ applies. This can be vividly seen by noting that the action of the pth power of $D^\star \circ D^\star$ on ϕ produces a p-nested even commutator of the type $[R, [R, [\cdots, [R, \phi]\cdots]]]$ which is, up to a factorial prefactor, the pth term in the well known formula for induced representations. Yet the identification $\phi = T$ is somewhat problematic, since in this case the underlying leaf carries infinite dimensions.

4. Algebraic formulation of a foliar structure

The two connections associated with the foliar complex can also be defined algebraically by means of infinitesimal changes in foliar frames. Let X denote an arbitrary vector field on M and let the $N_V \times N_U \left(= \dim V \times \dim U\right)$ frame fields $\boldsymbol{e}_a^A \otimes \bar{\boldsymbol{e}}_{\bar{a}}^{\bar{A}}$ span a basis for a leaf S at each point of M. Here and throughought the rest of this section small letters from the beginning of the alphabet label basis vectors, and the corresponding capital letters label their components. The differential of a fibre basis vector is clearly linear in that basis and given in terms of the connection one-form by the defining set of equations $d\boldsymbol{e}_a^A = \widetilde{\varphi}_a^b\left(X\right)\boldsymbol{e}_b^A := -\varphi_m\left(X\right)L^m{}_B^A\boldsymbol{e}_a^B$, where the index m (and later on, $\bar{n}$ as well) runs over the algebra, and the coefficients are evaluated on X. The inclusion of a second fibre within the context of a foliar complex can only be consistently done by considering the coefficients $\widetilde{\varphi}$ as tensor-valued in the counterpart representation space, $\left(d\boldsymbol{e}_a^A\right) \otimes \bar{\boldsymbol{e}}_{\bar{a}}^{\bar{A}} = \widetilde{\varphi}_{a\bar{B}}^{b\bar{A}}\left(X\right)\boldsymbol{e}_b^A \otimes \bar{\boldsymbol{e}}_{\bar{a}}^{\bar{B}} := -\varphi_{m\bar{n}}\left(X\right)L^m{}_B^A\boldsymbol{e}_a^B \otimes \bar{L}^{\bar{n}}{}_{\bar{B}}^{\bar{A}}\bar{\boldsymbol{e}}_{\bar{a}}^{\bar{B}}$, where the n_U $\bar{L}$'s are the generators of the counterpart group. Taking off the indices we write $\left(d\boldsymbol{e}\right) \otimes \bar{\boldsymbol{e}} = -\varphi\left(\boldsymbol{e} \otimes \bar{\boldsymbol{e}}\right)$, where it is understood that φ is a one-form $\in T^\star M$ whose action on a leaf basis $\left(\boldsymbol{e} \otimes \bar{\boldsymbol{e}}\right)$ is carried out via an appropriate representation of the generators. Now, for the computation of the differential of the entire basis, two distinct connections must be (uniquely) introduced:

$$
\begin{aligned}
d\left(\boldsymbol{e}_a^A \otimes \bar{\boldsymbol{e}}_{\bar{a}}^{\bar{A}}\right) &= \left(d\boldsymbol{e}_a^A\right) \otimes \bar{\boldsymbol{e}}_{\bar{a}}^{\bar{A}} + \boldsymbol{e}_a^A \otimes \left(d\bar{\boldsymbol{e}}_{\bar{a}}^{\bar{A}}\right) = \widetilde{\varphi}_{a\bar{B}}^{b\bar{A}}\left(X\right)\boldsymbol{e}_b^A \otimes \bar{\boldsymbol{e}}_{\bar{a}}^{\bar{B}} + \widetilde{\omega}_{\bar{a}B}^{\bar{b}A}\left(X\right)\boldsymbol{e}_a^B \otimes \bar{\boldsymbol{e}}_{\bar{b}}^{\bar{A}} \\
&:= -\varphi_{m\bar{n}}\left(X\right)L^m{}_B^A\boldsymbol{e}_a^B \otimes \bar{L}^{\bar{n}}{}_{\bar{B}}^{\bar{A}}\bar{\boldsymbol{e}}_{\bar{a}}^{\bar{B}} - \omega_{m\bar{n}}\left(X\right)L^m{}_B^A\boldsymbol{e}_a^B \otimes \bar{L}^{\bar{n}}{}_{\bar{B}}^{\bar{A}}\bar{\boldsymbol{e}}_{\bar{a}}^{\bar{B}} \qquad (7)
\end{aligned}
$$

that is, $d\left(\boldsymbol{e} \otimes \bar{\boldsymbol{e}}\right) = -\varphi\left(\boldsymbol{e} \otimes \bar{\boldsymbol{e}}\right) - \omega\left(\boldsymbol{e} \otimes \bar{\boldsymbol{e}}\right)$. Despite the structural resemblance in the above two terms, the coefficients of φ and ω are indeed different entities; whereas φ is uniquely determined by the set of equations which relates it to $\widetilde{\varphi}$, which is induced by the gauge group G_V, ω is determined by a different set of equations, namely, the one which relates it to $\widetilde{\omega}$, induced by the gauge group G_U. Of course, the transformation laws of (3) are solutions of definition (7); but furthermore, consistency requires the compatibility of (7) with a *combined* G_V–G_U $\left(= G_U\text{–}G_V\right)$ gauge transformation on the entire leaf. Let us see that this is indeed the case. An action of $\left(v \times u\right) \in G_V \times G_U$ to the left of the leaf basis reads: $\left(v \times u\right)\left(\boldsymbol{e} \otimes \bar{\boldsymbol{e}}\right) := \left(v\boldsymbol{e} \otimes u\bar{\boldsymbol{e}}\right)$. Now, for the differential of a transformed basis we get

$$
d\left(v\boldsymbol{e} \otimes u\bar{\boldsymbol{e}}\right) = \left(dv\right)\boldsymbol{e} \otimes u\bar{\boldsymbol{e}} - v \circ \varphi'\left(\boldsymbol{e} \otimes u\bar{\boldsymbol{e}}\right) + v\boldsymbol{e} \otimes \left(du\right)\bar{\boldsymbol{e}} - \left(v \times u\right) \circ \omega\left(\boldsymbol{e} \otimes \bar{\boldsymbol{e}}\right) \qquad (8)
$$

where φ' (which is φ transformed by G_U) is still left to be determined. On the other hand, the transformed left-hand side of (7) reads

$$G_V \times G_U : -\varphi\Big(e \otimes \bar{e}\Big) - \omega\Big(e \otimes \bar{e}\Big) \mapsto -\varphi''\Big(ve \otimes u\bar{e}\Big) - \omega''\Big(ve \otimes u\bar{e}\Big).$$

Following the laws of (3) we set

$$-\varphi'' = -u\Big(v\varphi v^{-1} + vdv^{-1}\Big)u^{-1} \qquad \text{and} \qquad -\omega'' = -v\Big(u\omega u^{-1} + udu^{-1}\Big)v^{-1}$$

from which

$$\begin{aligned} -\varphi''\Big(ve \otimes u\bar{e}\Big) &= -\Big(u \times v\Big) \circ \varphi\Big(e \otimes \bar{e}\Big) + \Big(dv\Big)e \otimes u\bar{e} \\ -\omega''\Big(ve \otimes u\bar{e}\Big) &= -\Big(v \times u\Big) \circ \omega\Big(e \otimes \bar{e}\Big) + ve \otimes \Big(du\Big)\bar{e}. \end{aligned} \tag{9}$$

Now identifying the two left-hand sides of (9) with the left-hand side of (8) implies $\varphi'\Big(e \otimes u\bar{e}\Big) = u \circ \varphi\Big(e \otimes \bar{e}\Big)$, in precise agreement with $G_U : \varphi \mapsto u\varphi u^{-1}$.

Example. In a fundamental representation, the indices which label the basis vectors are of the same type as those which label their components. In particular, for the classical groups (and their related sub-structures) the algebra also employs the same type of indices. Consider a Whitney product of two tangent bundles, both associated with the same manifold M. The arena of leaf-valued forms lies in

$$\bigcup_{x\in M}\left(\bigoplus_{p=0}^{n}\bigwedge^p T_x^\star M\right)\left(\bigotimes_{\alpha=1}^{2} T_x^{\left(\alpha\right)} M\right).$$

The foliar structure is induced by two independent $\mathrm{GL}\Big(n, R\Big)$ groups whose generating elements are realized on T_xM via the defining representation $(L^a_b)^B_A = \delta^a_A\delta^B_b$, from which commutation $\Big(-\Big)$ and anticommutation $\Big(+\Big)$ relations are easily computed: $[L^a_b, L^c_d]_\mp = \Big(\delta^c_b\delta^a_e\delta^f_d \mp \delta^a_d\delta^c_e\delta^f_b\Big)L^e_f$. The realizations above imply that φ and ω both acquire a particularly simple form:

$$\varphi^{a\bar{a}}_{b\bar{b}}(L^b_a \otimes L^{\bar{b}}_{\bar{a}})^{A\bar{A}}_{B\bar{B}} = \varphi^{A\bar{A}}_{B\bar{B}} = -\tilde{\varphi}^{A\bar{A}}_{B\bar{B}} \qquad \text{and} \qquad \omega^{a\bar{a}}_{b\bar{b}}(L^b_a \otimes L^{\bar{b}}_{\bar{a}})^{A\bar{A}}_{B\bar{B}} = \omega^{A\bar{A}}_{B\bar{B}} = -\tilde{\omega}^{A\bar{A}}_{B\bar{B}}.$$

The classical gauge thus identifies the notion of parallelism, which is based on the concept of a tilde connection, with that of horizontality, based on Yang–Mills connections with values in Lie algebras. Consequently

$$R^{A\bar{A}}_{B\bar{B}} = d\varphi^{A\bar{A}}_{B\bar{B}} + \varphi^{A\bar{A}}_{C\bar{C}} \wedge \varphi^{C\bar{C}}_{B\bar{B}} + \varphi^{A\bar{A}}_{C\bar{C}} \wedge \omega^{C\bar{C}}_{B\bar{B}} + \omega^{A\bar{A}}_{C\bar{C}} \wedge \varphi^{C\bar{C}}_{B\bar{B}} + \omega^{A\bar{A}}_{C\bar{C}} \wedge \omega^{C\bar{C}}_{B\bar{B}} + d\omega^{A\bar{A}}_{B\bar{B}}$$

which can be viewed as a generalization of the expression for the components of a single-structure curvature, given by $R^A_B = d\varphi^A_B + \varphi^A_C \wedge \varphi^C_B$. Indeed, the *folium curvature* of a smooth manifold M, is given by a $\Big(2, 2\Big)$-tensor two-form on M.

On the basis of the above viewpoints, one is naturally led to infer that the curvature R_{FC} of a foliar complex correlates between the two 'half-linear' primordial curvatures $d\varphi + \varphi \wedge \varphi$ and $d\omega + \omega \wedge \omega$, remnants of the single-fibre bundles VM and UM, via the two coupled terms $\omega \wedge \varphi$ and $\varphi \wedge \omega$. If one transports a *leaf* horizontally along a close path on the base-space (in non-flat directions), one measures a curvature which is different from that obtained by taking the sum of single fibre treks, as one usually does by the process of trivial splicing. In fact, the foliar complex can be considered as a *unifying infrastructure* within which two gauges are composed into a single structure whose curvature $R = d\Big(\varphi + \omega\Big) + \Big(\varphi + \omega\Big) \wedge \Big(\varphi + \omega\Big)$ is made of a single connection $\Big(\varphi + \omega\Big)$ with values in a space product of two Lie algebras, and which satisfies a two-group implementation of a single transformation law, $G_V : \Big(\varphi + \omega\Big) \mapsto v\Big(\varphi + \omega + d\Big)v^{-1}$, and $G_U : \Big(\varphi + \omega\Big) \mapsto u\Big(\varphi + \omega + d\Big)u^{-1}$. In this context, the curvature coefficients are given by the more familiar form, $R^{c\bar{c}} = d\Big(\varphi + \omega\Big)^{c\bar{c}} + \frac{1}{2}C_{VU}^{a\bar{a}b\bar{b}c\bar{c}}\Big(\varphi + \omega\Big)_{a\bar{a}} \wedge \Big(\varphi + \omega\Big)_{b\bar{b}}$ with the single-group structure constants now being replaced by the foliar ones.

We conclude this section by establishing the exact relations between absolute differentials and covariant exterior derivatives. To this end we note that if $\{\boldsymbol{e}_\alpha\}$ stands for a set of frame fields of VM and $g^{\alpha\beta} := g_{\alpha\beta}^{-1}$, where $g_{\alpha\beta} := \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$ is a local metric on the fibre, we have $\boldsymbol{e}^\alpha := g^{\alpha\beta}\boldsymbol{e}_\beta$, from which $\boldsymbol{e}^\alpha \cdot \boldsymbol{e}_\gamma = g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha{}_\gamma$. In contrast, however, the coframe field monomials of $V^\star M$ are generated by the set of functionals $\{\widetilde{\boldsymbol{e}}^\alpha\}$ satisfying $\widetilde{\boldsymbol{e}}^\alpha\Big(\boldsymbol{e}_\beta\Big) = \delta^\alpha{}_\beta$. Therefore, for any $\boldsymbol{e} \in VM$, we have $\Big(d\boldsymbol{e}^a\Big) \cdot \boldsymbol{e}_b = -\boldsymbol{e}^a \cdot \Big(d\boldsymbol{e}_b\Big) = +\boldsymbol{e}^a \cdot \varphi\Big(X\Big)\boldsymbol{e}_b \Rightarrow d\boldsymbol{e}^a = +\varphi\Big(X\Big)\boldsymbol{e}^a$, and similarly, for any $\bar{\boldsymbol{e}} \in UM$, $d\bar{\boldsymbol{e}}^{\bar{a}} = +\omega\Big(X\Big)\bar{\boldsymbol{e}}^{\bar{a}}$, but the coframe functionals of $V^\star M$ and $U^\star M$ are all annihilated by d because $d\widetilde{\boldsymbol{e}} = \Big(\partial\widetilde{\boldsymbol{e}}/\partial\boldsymbol{e}\Big)d\boldsymbol{e} = 0$. Keeping an open eye on the order in which terms are arranged in an expression, it immediately follows that $d\Big(\boldsymbol{e}^a \otimes \bar{\boldsymbol{e}}^{\bar{a}}\psi_{a\bar{a}}\Big) \equiv \boldsymbol{e}^a \otimes \bar{\boldsymbol{e}}^{\bar{a}}\Big(D\psi\Big)_{a\bar{a}}$ for vector-valued forms, whereas $d\big(\boldsymbol{e}^a \otimes \bar{\boldsymbol{e}}^{\bar{a}}\psi_{a\bar{a}}^{b\bar{b}} \otimes \boldsymbol{e}_b \otimes \bar{\boldsymbol{e}}_{\bar{b}}\big) \equiv \boldsymbol{e}^a \otimes \bar{\boldsymbol{e}}^{\bar{a}}\Big(D^\star\psi\Big)_{a\bar{a}}^{b\bar{b}} \otimes \boldsymbol{e}_b \otimes \bar{\boldsymbol{e}}_{\bar{b}}$ for tensor-valued ones. Being a little careless with standards of rigour (by identifying differentials taken in different directions) we may take advantage of the 'failure' of $d \circ d$ to annihilate vector fields and further construct the curvature and other objects of interest by successive applications of d.

5. Global rescaling and local translations in the spaces of connections

We next observe that the foliar complex construction is defined up to multiplicative constants added to the foliar pair of connections; the substitutions

$$\varphi \to a^{-1}\varphi \qquad \omega \to b^{-1}\omega \qquad a, b \in K \tag{10}$$

are compatible with the gauge transformation laws of (3) and therefore maintain covariance at any level. One may, for example, normalize the connections by choosing $a = \int \mathcal{D}[\varphi]$ and $b = \int \mathcal{D}[\omega]$ where the functional integrations are taken in connection space (modulo gauge transformation) and the $\mathcal{D}$'s are the appropriate measures. Nonetheless, each single term in the expressions for the covariant exterior derivatives and the curvature is now weighted differently. Therefore, a global rescaling in each space of connections tunes the curvature, thereby reproducing three potential 'self-interaction' couplings (namely, gluon–gluon): a^{-2}, b^{-2} and $a^{-1}b^{-1}$, where of particular interest is the third one which weights *cross-gauge* interactions.

Consider next a single-fibre gauge structure, and let the connection ω translate according to $\omega \to \omega + \Omega$, where big Ω is an x-dependent 'co-frame' one-form taking its values in the same Lie algebra as ω. By construction, Ω cannot be gauged out of the bundle (like a pure gauge) and therefore ω and $\omega + \Omega$ can never be connected by a gauge transformation. Being of a completely horizontal nature, the translation by Ω generates bijections between equivalence classes in the coset space of connections modulo gauge transformations. Now, while ω obviously still transforms properly after being shifted, the curvature no longer stays invariant. One is therefore led to introduce the notion of a 'vertical' Grassmann algebra, defined over the group manifold and graded by an appropriate co-boundary operator δ with respect to which Ω is considered as one-form as well. Over the extended arena of forms, where we allow everything to depend on all possible degrees of freedom, δ anticommutes with d. Exploiting the operatorial definition for the curvature by successively applying the covariant exterior derivative to an arbitrary test form of the extended Grassmann space, we find after some calculation

$$\begin{aligned} D^\star \circ D^\star \psi &= D^\star \left(d\psi + \delta\psi + \left(\omega + \Omega\right) \wedge \psi + \psi \wedge \left(\omega + \Omega\right)\right) \\ &= [R, \psi] + \left[D^\star \Omega, \psi\right] + [\Omega \wedge \Omega, \psi] + \left[\delta\left(\omega + \Omega\right), \psi\right]. \end{aligned} \tag{11}$$

The vanishing of the extra three commutators in (11) uniquely implies $\delta\omega = -D^\star\Omega$ and $\delta\Omega = -\Omega \wedge \Omega$ (uniqueness: a consequence of compatibility with the gradings) whereas, in the absence of any other conditional terms, the variation of ψ is left totally undetermined. One can easily check that a squared variation $\delta\delta$ vanishes on both ω and Ω. Two subsidiary comments are in order: First, the indifference of the curvature to horizontal translations in the space of connections, induced by the above structure, just manifests a known ambiguity due to Wu and Yang, where two gauge-disconnected non-Abelian connections give rise to the same curvature. Second, the coefficients Ω^a (and later on $\Omega^{a\bar{a}}$ and $\Phi^{a\bar{a}}$ as well), being differential forms, vanish upon exterior squaring and can therefore be identified with the *ghosts* of the gauge. The following digression concludes our single-structure prelude.

Digression. According to the conventional geometric approach to ghost sectors and BRST co-homology [2], one enlarges the classical Grassmann bundle to include the vertical space spanned by the group angles: pick a local coordinate frame $x \in M$, $\phi\left(x\right) \in G$, and put $\mathring{\omega} = \omega^a_\mu\left(x, \phi\left(x\right)\right) L_a dx^\mu + C^a_b\left(x, \phi\left(x\right)\right) L_a \delta\phi^b\left(x\right)$, where the L_a's $\in \mathrm{Lie}\, G$ and the C_b's stand for the vertical components of the connection. Note: we now deal with an extended base in which for any $g \in G$, $\mathring{\omega} \mapsto g\left(\mathring{\omega} + d + \delta\right)g^{-1}$. By a slight abuse of notation, the corresponding two co-boundary operators are realized by $d := dx^\mu \partial/\partial x^\mu$ and $\delta := d\phi^a \partial/\partial\phi^a$, where the realizations in particular imply $d \circ d = \delta \circ \delta = d \circ \delta + \delta \circ d = 0$, and moreover, $\delta\phi^a = d\phi^a$. As a consequence of the above extension, an expansion of the corresponding curvature results in four terms, $R_{\text{extended}} = R_{\text{hh}} + R_{\text{hv}} + R_{\text{vh}} + R_{\text{vv}}$, where R_{hh} is a purely horizontal form, R_{vv} is a purely vertical one, and R_{hv}, R_{vh} are forms of a mixed basis. Then, the variations of ω and Ω with respect to δ follow by requiring flatness in vertical directions, namely, by imposing $R_{\text{hv}} = R_{\text{vh}} = R_{\text{vv}} = 0$. In this context, the gauge sector is said to possess an internal BRST structure. Now, G is a smooth manifold $\Rightarrow$ $d\phi^a = \left(\partial\phi^a/\partial x^\mu\right)dx^\mu \Rightarrow \mathring{\omega}$ employs base-space components only. This, in turn, exactly relates our Ω^a's to the ghosts of the above description, namely, $\Omega^a \equiv C^a_b(\partial\phi^b/\partial x^\mu)dx^\mu$, where $\mathring{\omega}$ is just our original ω that has been shifted by Ω. The idea is easily extended to forms of arbitrary degree where the ghost index counts the number of $\left(\partial\phi/\partial x\right)$'s occurring in a base space implementation, *not* the vertical gradation. It is therefore why a connection ω, as opposed to a shift Ω, carries ghost index zero.

Let us now get back to foliar complexes. In this case, each of the two connection forms ω and φ is shifted in its own coset space by a corresponding 'co-frame' one-form (say, big Ω and big Φ respectively), taking values in Lie $G_V \otimes$ Lie G_U. Under these circumstances, both shifted terms transform precisely according to equations (3). In order to fix the curvature, we let the connections and the shifted terms depend on vertical variables as well. This time, however, there are two sets of them, corresponding to the two group manifolds spanned by G_V and G_U. Each of the associated Grassmann spaces is graded by its own co-boundary operator (denoted respectively by δ and $\bar{\delta}$) with respect to which Ω and Φ are considered as one-forms. One now encounters two ghost indexes: one is generated by δ, the other is generated by $\bar{\delta}$; a single quantum of the former is carried by Ω, a single quantum of the latter, by Φ. And of course, $d\delta + \delta d = d\bar{\delta} + \bar{\delta} d = \delta\bar{\delta} + \bar{\delta}\delta = 0$. Making use of the operatorial definition for the curvature with respect to a base space of $n \times n_V \times n_U$ dimensions, we find

$$\begin{aligned} D^\star \circ D^\star\psi = [R, \psi] &+ \left[D^\star\left(\Omega + \Phi\right), \psi\right] + \left[\left(\Omega + \Phi\right) \wedge \left(\Omega + \Phi\right), \psi\right] \\ &+ \left[\delta\left(\omega + \Omega + \varphi + \Phi\right), \psi\right] + \left[\bar{\delta}\left(\omega + \Omega + \varphi + \Phi\right), \psi\right]. \end{aligned} \tag{12}$$

Once more, we set the extra four terms to vanish by equating the terms of equal ghost index. The resulted variation laws read

$$\begin{aligned} &\delta\left(\omega + \varphi\right) = -D^\star\Omega \qquad &&\bar{\delta}\left(\omega + \varphi\right) = -D^\star\Phi \\ &\delta\Omega = -\Omega \wedge \Omega &&\bar{\delta}\Omega = -\Phi \wedge \Omega \\ &\delta\Phi = -\Omega \wedge \Phi &&\bar{\delta}\Phi = -\Phi \wedge \Phi. \end{aligned} \tag{13}$$

We see that we cannot derive separate variation laws for φ and ω. This, however, is totally compatible with our previous claim where we argued that $\varphi + \omega$ can be regarded as a single connection of a single gauge structure with two gauge groups; the BRST variation sees only one gauge connection.

It is manifestly clear that equation (13) is completely invariant with respect to a duality transformation, $\delta \leftrightarrow \bar{\delta}$ and $\Omega \leftrightarrow \Phi$. In addition, one easily verifies that the two squared variations, $\delta\delta$ and $\bar{\delta}\bar{\delta}$, vanish on Ω, Φ and $\omega + \varphi$. Now, the information about how ghosts vary with respect to the co-boundary operators of the counterpart Grassmann space can be compactly described by $\delta\Phi + \bar{\delta}\Omega + [\Omega, \Phi] = 0$, see equation (13). Guided by such a duality-invariant relation, let us introduce $B := \delta\Phi = -\Omega \wedge \Phi$, by construction δ-exact, an entity whose coefficients in Lie $G_V \otimes$ Lie G_U are commuting differential forms. Then, due to the nilpotency of δ and due to (13),

$$\delta B = 0 \qquad \bar{\delta} B = -[\Phi, B] \Rightarrow \bar{\delta}\bar{\delta} B = 0. \tag{14}$$

The structural and variation properties of B suggest identifying it with the so-called B-field in the context of the BRST mechanism. However, in contrast to previous treatments [2, 3], here it appears not as additional degrees of freedom, but rather as a simple composition of already existing ones, namely, a ghost–ghost exterior product. Had we started, however, with an alternative assignment, $\bar{B} := \bar{\delta}\Omega = -\Phi \wedge \Omega$, we would have obtained the same results, only within the dual description. It therefore makes sense to identify the dual $\bar{B}$-field with the antifield of B. Consider for a moment the case of a unitary structure: being just shifts between Hermitian fields, Φ and Ω are themselves Hermitian, and we have $B^\dagger = -\bar{B}$. Following this line of reasoning, Φ and Ω appear as antifields of one another. We can therefore pick one of them, no matter which, to play the role of an antighost. These interpretations, together with the derived variation laws above, are seen to cover the entire BRST structure of a foliar complex, all by pure geometrical means. Apparently, it coincides with the ghost–antighost super algebra one usually associates with local gauge theories.

6. Splices which admit contractable pieces

Let us now soften the constraints for closed anticommutability in an appropriate representation and include identity elements as well. Namely, the generators of G_V, while carrying representations in V, satisfy $\{L^a, L^b\} = (2/N_V)d_V^{ab} + g_V^{abc}L_c$, where $N_V = \dim V$, $d_V^{ab} = \mathrm{Tr}(L^a L^b)$ and similar relations hold for the $\bar{L}$'s of G_U with the corresponding d_U's, and $N_U = \dim U$. (Note: this is exactly the case of two $SU\left(n\right)$ structures in their fundamental representation). Now, in order to comply with the appearance of the identity, one first prolongs the vector spaces spanned by the generating algebras by adding $L_0 = \bar{L}_0 := I$, and then extends the closure formulae (4) to include (we use $\alpha, \beta, \ldots = 0, 1, \ldots, n_V$ and $\bar{\alpha}, \bar{\beta}, \ldots = 0, 1, \ldots, n_U$)

$$\begin{aligned} g_V^{\alpha\beta 0} &= \frac{2}{N_V} d_V^{\alpha\beta} \qquad & g_V^{\alpha 0\gamma} &= 2\delta^{\alpha\gamma} \\ g_U^{\bar{\alpha}\bar{\beta}0} &= \frac{2}{N_U} d_U^{\bar{\alpha}\bar{\beta}} \qquad & g_U^{\bar{\alpha}0\bar{\gamma}} &= 2\delta^{\bar{\alpha}\bar{\gamma}} \end{aligned} \tag{15}$$

where for the f's we put $f_V^{\alpha\beta\gamma} = 0$ if *any* of the indices α, β or γ is zero, and the same for $f_U^{\bar{\alpha}\bar{\beta}\bar{\gamma}}$ with $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$. Note that the $d^{\alpha\beta}$'s and the $d^{\bar{\alpha}\bar{\beta}}$'s are now interpreted as Cartan metrics on the unitals (unital = a Lie algebra + the unity). Let us extract the consequences of this generalization; the original connection one-forms, in principle, are still valued in $\mathrm{Lie}\, G_V \otimes \mathrm{Lie}\, G_U$. Introduce, however, an extra pair of scalar one-forms $\left(\varphi_{00} L^0 \otimes \bar{L}^0, \omega_{00} L^0 \otimes \bar{L}^0\right)$, an extra pair of single-fibre connections $\left(\varphi_{a0} L^a \otimes \bar{L}^0, \omega_{0\bar{a}} L^0 \otimes \bar{L}^{\bar{a}}\right)$ and a pair of single-fibre tensor one-forms $\left(\varphi_{0\bar{a}} L^0 \otimes \bar{L}^{\bar{a}}, \omega_{a0} L^a \otimes \bar{L}^0\right)$ and put $\widehat{\omega} = \omega_{\alpha\bar{\alpha}} L^\alpha \otimes \bar{L}^{\bar{\alpha}}$, and $\widehat{\varphi} = \varphi_{\alpha\bar{\alpha}} L^\alpha \otimes \bar{L}^{\bar{\alpha}}$.

Comment. Namely, $\varphi_{a0} L^a \otimes \bar{L}^0$ is a connection form on VM, but $\varphi_{0\bar{a}} L^0 \otimes \bar{L}^{\bar{a}}$ is a rank-2 G_U-tensor form on UM; $\omega_{0\bar{a}} L^0 \otimes \bar{L}^{\bar{a}}$ is a connection form on UM, but $\omega_{a0} L^a \otimes \bar{L}^0$ is a rank-2 G_V-tensor form on VM.

Now, since $\widehat{\varphi}$ and $\widehat{\omega}$ are actually valued in $\left(\mathrm{Lie}\, G_V + I_V\right) \otimes \left(\mathrm{Lie}\, G_U + I_U\right)$, any symmetric product of two 'hats', like $\widehat{\omega} \wedge \widehat{\omega}$ or $[\widehat{\varphi}, \widehat{\omega}]$, necessarily involves a structural classification according to $\mathrm{Lie}\, G_V \otimes \mathrm{Lie}\, G_U + \mathrm{Lie}\, G_V \otimes I_U + I_V \otimes \mathrm{Lie}\, G_U + I_V \otimes I_U$. We note that the trivial part of the splice is present via $\mathrm{Lie}\left(G_V \times G_U\right)$. Tracing the recipes given by the proposition of section 3, we find the following expression for the total 'curvature' two-form $\widehat{R}$:

$$\widehat{R} := 2 \times d\widehat{\omega} + \widehat{\omega} \wedge \widehat{\omega} + \left(\widehat{\omega} \wedge \widehat{\varphi} + \widehat{\varphi} \wedge \widehat{\omega}\right) + \widehat{\varphi} \wedge \widehat{\varphi} + 2 \times d\widehat{\varphi}$$

$$= \sum_{\varpi, \varpi' = \widehat{\varphi}, \widehat{\omega}} \left(2d\varpi_{\gamma\bar{\gamma}} + \tfrac{1}{2}\widehat{C}_{VU}^{\alpha\bar{\alpha}\beta\bar{\beta}}{}_{\gamma\bar{\gamma}}\, \varpi_{\alpha\bar{\alpha}} \wedge \varpi'_{\beta\bar{\beta}}\right) L^{\gamma} \otimes \bar{L}^{\bar{\gamma}} \tag{16}$$

with $\widehat{C}_{VU}^{\alpha\bar{\alpha}\beta\bar{\beta}\gamma\bar{\gamma}} = \frac{1}{2}\left(f_V^{\alpha\beta\gamma} g_U^{\bar{\alpha}\bar{\beta}\bar{\gamma}} + f_U^{\bar{\alpha}\bar{\beta}\bar{\gamma}} g_V^{\alpha\beta\gamma}\right) + f_V^{\alpha\beta\gamma} f_U^{\bar{\alpha}\bar{\beta}\bar{\gamma}}$. (The exact differentials above appear twice just in order to account for linearity; we shall discuss this soon.) The structural classification of (16) is now dictated by the $\gamma\bar{\gamma}$-pairs: the $\widehat{C}_{VU}$-attached 00-pair obviously vanishes (because each term in $\widehat{C}_{VU}$ has an f representative), the $c0$-pairs and the $0\bar{c}$-pairs each correspond to a single-fibre structure, whereas the $c\bar{c}$-pairs correspond to a foliar-like structure.

One should, however, be very cautious with gauge interpretations. Almost none of these decomposed terms vary linearly with respect to gauge transformations. What then are the transformation properties of $\widehat{R}$? Before answering this question, let us adopt the following short-hand notation:

$$\widehat{\varphi} = \varphi_{00} L^0 \otimes \bar{L}^0 + \varphi_{a0} L^a \otimes \bar{L}^0 + \varphi_{0\bar{a}} L^0 \otimes \bar{L}^{\bar{a}} + \varphi_{a\bar{a}} L^a \otimes \bar{L}^{\bar{a}} := \varphi^{00} + \varphi^{10} + \varphi^{01} + \varphi^{11}$$

and the same for $\widehat{\omega}$. Then, for example, φ^{10} is a G_U-scalar connection one-form on VM, $d\varphi^{10} + \varphi^{10} \wedge \varphi^{10}$ is a VM-curvature element

$$\in R^{1010} = d\varphi^{10} + \varphi^{10} \wedge \varphi^{10} + \varphi^{10} \wedge \omega^{10} + \omega^{10} \wedge \varphi^{10} + \omega^{10} \wedge \omega^{10} + d\omega^{10}$$

whereas $\varphi^{01} \wedge \varphi^{01}$ is a pure G_U-tensor

$$\in R^{0101} = d\varphi^{01} + \varphi^{01} \wedge \varphi^{01} + \varphi^{01} \wedge \omega^{01} + \omega^{01} \wedge \varphi^{01} + \omega^{01} \wedge \omega^{01} + d\omega^{01}$$

etc. In addition we put

$$R^{1111} = d\varphi^{11} + \varphi^{11} \wedge \varphi^{11} + \varphi^{11} \wedge \omega^{11} + \omega^{11} \wedge \varphi^{11} + \omega^{11} \wedge \omega^{11} + d\omega^{11}$$

but note that this object is of a mixed algebraic structure because the wedge operation now activates identity elements as well; it however transforms linearly with respect to both gauge groups. Now, as the quantity $\widehat{R}$ is totally symmetric with respect to its two arguments, φ and ω, all of its wedge products that involve scalar forms (there are 36 of them) vanish identically, and moreover, any of the four symmetric pairs $\left(1001\right)$ and $\left(0110\right)$ vanish as well because each wedge product in such a pair consists of two commuting terms, each of which is living in a different representation space. Therefore, and with the above conventions at our disposal, $\widehat{R}$ decomposes into

$$\widehat{R} = R^{1010} + d\varphi^{10} + d\omega^{10} + R^{0101} + d\varphi^{01} + d\omega^{01} + R^{1111} + d\varphi^{11} + d\omega^{11}$$
$$+ 2d\omega^{00} + 2d\varphi^{00} + \left[\varphi^{10} + \varphi^{01} + \omega^{10} + \omega^{01}, \varphi^{11} + \omega^{11}\right] \tag{17}$$

(the last term is a graded commutator). It is straightforward, and not too cumbersome, to verify the following gauge transformation properties of $\widehat{R}$:

$$G_V : \begin{cases} \left.\begin{array}{c} R^{1010} + R^{1111} + d\varphi^{10} + d\omega^{10} + d\varphi^{11} + d\omega^{11} \\ + \left[\varphi^{10} + \varphi^{01} + \omega^{10} + \omega^{01}, \varphi^{11} + \omega^{11}\right] \end{array}\right\} \implies G_V\text{-tensor} \\ R^{0101} + 2d\varphi^{00} + d\varphi^{01} + 2d\omega^{00} + d\omega^{01} \implies G_V\text{-scalar} \end{cases} \tag{18}$$

$$G_U : \begin{cases} \left. \begin{array}{c} R^{0101} + R^{1111} + d\varphi^{01} + d\omega^{01} + d\varphi^{11} + d\omega^{11} \\ + \left[\varphi^{10} + \varphi^{01} + \omega^{10} + \omega^{01}, \varphi^{11} + \omega^{11}\right] \end{array} \right\} \implies G_U\text{-tensor} \\ R^{1010} + 2d\varphi^{00} + d\varphi^{10} + 2d\omega^{00} + d\omega^{10} \implies G_U\text{-scalar}. \end{cases} \tag{19}$$

Comment. In fact, $\left(d\varphi^{10} + d\varphi^{11} + d\omega^{10} + d\omega^{11}\right) + \left[\varphi^{10} + \varphi^{01} + \omega^{10} + \omega^{01}, \varphi^{11} + \omega^{11}\right]$, R^{1010} and R^{1111} all behave as independent G_V-covariant quantities. In the same manner, $\left(d\omega^{01} + d\varphi^{11} + d\omega^{01} + d\omega^{11}\right) + \left[\varphi^{10} + \varphi^{01} + \omega^{10} + \omega^{01}, \varphi^{11} + \omega^{11}\right]$, R^{0101} and R^{1111}, all behave as independent G_U-covariant quantities.

In conclusion, $\widehat{R}$ possesses simple but non-trivial transformation properties. In particular, it admits terms that behave as scalars under various group actions, each one at a time. One should not, however, be too much surprised. One now deals with structures that are induced by central extensions of Lie algebras rather than the Lie algebras themselves. In the former case, as opposed to the latter one, one starts with a 'curvature' which inherently contains scalar pieces (with respect to one gauge group or the other, or both) and therefore one ends up with scalar terms. This fact, in general, leads to the lack of covariance for the case where sectors are classified according to their algebraic structure, where there is only one exception: there are two autonomic decoupled sectors which are linear, and whose definite algebraic structure is completely preserved by the transformations, namely, the two single fibre bundles whose composition is just given by trivial means:

$$R^{1010} = \sum_{\varpi, \varpi' = \omega, \varphi} \left(d\varpi^{c0} + \tfrac{1}{2} f_V^{abc} \varpi_{a0} \wedge \varpi'_{b0}\right) L_c \otimes I_U \quad \in VM \tag{20}$$

$$R^{0101} = \sum_{\varpi, \varpi' = \omega, \varphi} \left(d\varpi^{0\bar{c}} + \tfrac{1}{2} f_U^{\bar{a}\bar{b}\bar{c}} \varpi_{0\bar{a}} \wedge \varpi'_{0\bar{b}}\right) I_V \otimes \bar{L}_{\bar{c}} \quad \in UM. \tag{21}$$

These two pieces close on themselves and therefore can be treated independently of everything else. In this case, single-structure bundle operators will certainly do. Otherwise, if everything is taken into account, one has to utilize modified bundle operators which generalize the foliar complex covariant exterior derivatives. In particular, the appropriate expression for a covariant exterior derivative of vector-valued leaf forms reads: $\widehat{D}\psi := 2d\psi + \left(\widehat{\varphi} + \widehat{\omega}\right) \wedge \psi$ while that of tensor-valued leaf forms, $\widehat{D}^\star\psi := 2d\psi + \left(\widehat{\varphi} + \widehat{\omega}\right) \wedge \psi + \left(-1\right)^{p+1} \psi \wedge \left(\widehat{\varphi} + \widehat{\omega}\right)$ where $p = \deg\psi$. One then treks familiar pathways, first by reconstructing the 'curvature' via $\left[\widehat{D}, \widehat{D}\right]\psi = 2\widehat{R} \wedge \psi$, or alternatively via $\left[\widehat{D}^\star, \widehat{D}^\star\right]\psi := 2\left[\widehat{R}, \psi\right]$, and later constructing identities by successive applications of $\widehat{D}$ or $\widehat{D}^\star$. This explains why the scalar one-form terms that are present in $\widehat{D}$ and $\widehat{D}^\star$ are indeed integrated pieces of the model and cannot be dropped; they do not count for linearity but do for the sake of consistency and completeness.

7. Epilogue

We have seen that indeed nothing prevents us from generalizing the concept of a foliar complex, just by going to central extensions. Other extensions, however, are believed to be completely acceptable as long as anticommutability can be softly expressed in a closed form; that is, terms added to the algebra should not necessarily be proportional to the identity provided they can always be incorporated in the generalized form of (4) where the indices run over the prolonged vector space. As a result, one trades with the overall linearity and the pure foliar structure is lost. Instead one gets two autonomic, mutually transparent, single fibre structures embedded in wider frames in which some of the original foliar properties are still inherent.

It would seem that a generalization of the whole model to the case of composing *any* number of gauge structures is straightforward. For an r-fold composition with r gauge groups, introduce r connection one-forms with values in the product space of the corresponding r Lie algebras, each of which transforms as a connection with respect to its inducing gauge group and as a tensor with respect to all the other ones. The pure foliar complex is then read off from a curvature of the type

$$R = d\left(\varphi_1 + \cdots + \varphi_r\right) + \left(\varphi_1 + \cdots + \varphi_r\right) \wedge \left(\varphi_1 + \cdots + \varphi_r\right)$$

and the two types of covariant exterior derivatives of r-fold foliar forms are made of an r-connection one-form $\left(\varphi_1 + \cdots + \varphi_r\right)$. Such a large structure will admit an r-fold BRST symmetry with associated r ghosts. Once more, extensions at the level of the algebra break the pure foliar structure at the gain of two decoupled ones. The generalization idea, however, is still speculative and requires a closer examination.

Finally, we remark on the symmetric role played by the two gauge connections. In a quantum theory, since both multiplets involve the same representation under both groups, the symmetry becomes a source for degeneracies. In particular, states of the gauge sector are expected to mix with one another, and the phenomenology is no longer automatically compatible with the gauge structure, especially if some mechanism is added in order to break the original gauge symmetry.

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References

[1] As an example for a mathematical treatment related to this kind of algebras and representations, look at:
Sternberg S and Wolf J 1978 *Trans. Am. Math. Soc.* **238** 1

[2] See:
Baulieu L and Thierry-Mieg J 1982 *Nucl. Phys.* B **197** 477–508 and references therein
For a recent implementation of the method in non-commutative geometry see:
Lee C Y, Hwang D S and Ne'eman Y 1995 *Preprint* hep-th/9512215

[3] An exhaustive discussion on ghosts, anti-ghosts and BRST technology, not necessarily in a geometrical setting, is given in:
Nakanishi N and Ojima I 1990 *Covariant Operator Formalism of Gauge Theories and Quantum Gravity* (Singapore: World Scientific) pp 141–81